GOVERNING SINGULARITIES OF SCHUBERT VARIETIES

ALEXANDER WOO AND ALEXANDER YONG

ABSTRACT. We present a combinatorial and computational commutative algebra methodology for studying singularities of Schubert varieties of flag manifolds.

We define the combinatorial notion of *interval pattern avoidance*. For "reasonable" invariants $\mathcal{P}$ of singularities, we geometrically prove that this governs (1) the $\mathcal{P}$-locus of a Schubert variety, and (2) which Schubert varieties are globally not $\mathcal{P}$. The prototypical case is $\mathcal{P}$ ="singular"; *classical* pattern avoidance applies admirably for this choice [Lakshmibai-Sandhya'90], but is insufficient in general.

Our approach is analyzed for some common invariants, including Kazhdan-Lusztig polynomials, multiplicity, factoriality, and Gorensteinness, extending [Woo-Yong'04]; the description of the singular locus (which was independently proved by [Billey-Warrington '03], [Cortez '03], [Kassel-Lascoux-Reutenauer'03], [Manivel'01]) is also thus reinterpreted.

Our methods are amenable to computer experimentation, based on computing with *Kazhdan-Lusztig ideals* (a class of generalized determinantal ideals) using `Macaulay 2`. This feature is supplemented by a collection of open problems and conjectures.

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1. Overview

Let X_w be the Schubert variety of the complete flag variety $\mathrm{Flags}(\mathbb{C}^n)$ associated to a permutation w in the symmetric group S_n. One would like to describe and classify the singularities of X_w, as well as calculate invariants measuring their complexity. Solutions to such problems typically require techniques from and have important applications to geometry, representation theory, and associated combinatorics. Two recent surveys of some work in this area are [2, 9].

In this paper, we formulate a new combinatorial notion, a generalization of pattern avoidance we call *interval pattern avoidance*; we then use this idea to explore the singularities of Schubert varieties and their local invariants. The well-known Kazhdan-Lusztig polynomials show up as one local invariant, since their coefficients are the Betti numbers for the local intersection cohomology of the singularities. Indeed, a desire to further understand the combinatorics of Kazhdan-Lusztig polynomials is one source of motivation (and application) for this present work. However, there are many other noteworthy invariants of singularities, including factoriality, multiplicity, Gorensteinness, and Cohen-Macaulay type. We provide a uniform language to study such *semicontinuously stable* invariants, in an attempt to gain further insight into the singularities of Schubert varieties.

Informally, our principal thesis is that, for any of these "reasonable" local invariants of singularities of Schubert varieties, the question of where it assumes a particular value has a natural answer in terms of interval pattern avoidance. Our main result (Theorem 2.6) is a precise version of this assertion, together with a geometric explanation; proofs are given in Section 4.

The two most basic problems about singularities of specific Schubert varieties are

- Which X_w are singular?
- Where is X_w singular?

These questions have been answered. Following upon a geometric characterization by Ryan [31] and an earlier combinatorial characterization by Wolper [33], V. Lakshmibai and B. Sandhya [23] gave a simple characterization of singular Schubert varieties in terms of the combinatorial notion of *pattern avoidance*: X_w is smooth if and only if w avoids the patterns 3412 and 4231; see the definitions in Section 2. They also conjectured an explicit description for the singular locus of X_w in terms of pattern avoidance. This conjecture was solved independently by several groups [3, 11, 18, 26] around 2000. We reinterpret this result in terms of interval pattern avoidance.

Although much is known concerning general properties of singularities of Schubert varieties, little more is known for properties which not all Schubert varieties hold in common. Thanks to fundamental work during the 1980s including that of C. DeConcini and V. Lakshmibai [12], and S. Ramanan and A. Ramanathan [29, 30], we know that all Schubert varieties are Cohen-Macaulay and normal. In addition, A. Cortez [11] and L. Manivel [27] independently described the neighborhoods of *generic* points in the singular locus of a Schubert variety; understanding where and how these neighborhoods change at special points of the singular locus is a core theme in our present investigations. More recently in [35], we determined which Schubert varieties are Gorenstein; we introduced a notion there called Bruhat-restricted pattern avoidance, and interval pattern

avoidance is a further generalization which has the advantage of a geometric interpretation. We further pursue below the question of where a non-Gorenstein Schubert variety is Gorenstein, along with analogous questions for other local properties.

Analysis of specific questions from this viewpoint suggests new algebraic, geometric, and combinatorial questions and conjectures which we explore computationally using `Macaulay 2` [16]. This is explained in Sections 5 and 6. The associated commutative algebra is that of *Kazhdan-Lusztig ideals* (a class of ideals generalizing classical determinantal ideals); this commutative algebra is explicated in Section 3.

This report was written in part to help facilitate activities at the "Workshop on combinatorial and computational commutative algebra" (Fields Institute, July-August 2006). The workshop advances the use of computer algebra systems such as `Macaulay 2`. We wrote the `Macaulay 2` code `Schubsingular` as an exploratory complement to this paper.[1]

For simplicity, this paper focuses on the complete flag manifold in type A. This allows us to emphasize links to the traditional study of determinantal ideals in commutative algebra and avoid the need for terminology from the theory of algebraic groups. However, the ideas below can be extended with appropriate modifications to the other root systems and partial flag manifolds. Finally, although in this paper we work over $\mathbb{C}$ for convenience, our results are valid over any field $\Bbbk$ of any characteristic except as noted.

2. THE MAIN DEFINITIONS AND THEOREM

2.1. **Interval pattern avoidance.** Let $v \in S_m$ and $w \in S_n$ be two permutations, where $m \leq n$. We say v **embeds in** w if there exist indices $1 \leq \phi_1 < \phi_2 < \ldots < \phi_m \leq n$ such that $w(\phi_1), w(\phi_2), \ldots, w(\phi_m)$ are in the same relative order as $v(1), \ldots, v(m)$. In other words, we require that $w(\phi_j) < w(\phi_k)$ if and only if $v(j) < v(k)$. The permutation w is said to (classically) **avoid** v if no such embedding exists.

Recall that **Bruhat order**, which we denote by $\leq$, is the partial order on S_m defined by declaring that $u \leq v$ if $v = u(i \leftrightarrow j)$ and $\ell(v) > \ell(u)$, and taking the reflexive transitive closure. Here, $(i \leftrightarrow j)$ is the transposition switching positions i and j, and $\ell(v)$ denotes the **Coxeter length** of v, which is the length of any reduced expression for v as a product of simple reflections $s_i = (i \leftrightarrow i+1)$. Alternatively, $\ell(v)$ is also the number of **inversions** of v; inversions are pairs i, j with $1 \leq i < j \leq m$ such that $v(i) > v(j)$. Bruhat order is a partial order graded by Coxeter length.

We now give our main definition. Let $[u, v]$ and $[x, w]$ be intervals in the Bruhat orders on S_m and S_n respectively. We say that $[u, v]$ **(interval) pattern embeds in** $[x, w]$ if there is a common embedding $\Phi = (\phi_1, \ldots, \phi_m)$ of u into x and v into w, where the entries of x and w outside of Φ agree, and, furthermore, $[u, v]$ and $[x, w]$ are isomorphic as posets.

Note that the first two requirements already determine x given u, v, w, and Φ. To be precise, for a permutation $\sigma \in S_m$, let $\Phi(\sigma) \in S_n$ be the permutation where $\Phi(\sigma)(\phi_j) = w(\phi_{(v^{-1}\sigma)(j)})$, and $\Phi(\sigma)(k) = w(k)$ if $k \neq \phi_j$ for $1 \leq j \leq m$. Then the first two requirements force x to be equal to $\Phi(\sigma)$. Therefore, for convenience, we sometimes drop $x = \Phi(u)$ and say that $[u, v]$ embeds in w if $[u, v]$ embeds in $[\Phi(u), w]$. We also say simply that w **(interval) (pattern) avoids** $[u, v]$ if there are no interval pattern embeddings of $[u, v]$ into $[x, w]$ for any $x \leq w$.

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[1]Available at the authors' websites.

The following lemma gives a simple criterion for checking if a pattern embedding actually produces an interval pattern embedding. Its proof is simple and we omit it.

Lemma 2.1. *An embedding Φ of $[u, v]$ into $[\Phi(u), w]$ is an interval pattern embedding if and only if $\ell(v) - \ell(u) = \ell(w) - \ell(\Phi(u))$.*

Example 2.2. Let $v = 35142 = s_2 s_1 s_4 s_3 s_2 s_4$ and $u = 13524 = s_2 s_4 s_3$. Note $u \leq v$, and $\ell(v) - \ell(u) = 3$. Now let Φ be the embedding of v into $w = 589\underline{716}2\underline{3}4$ where the underlined positions indicate the embedding, which in symbols is given by $\phi_1 = 1, \phi_2 = 4, \phi_3 = 5, \phi_4 = 6, \phi_5 = 8$. Then $\Phi(u) = 189\underline{573}2\underline{6}4$. The reader can check that $\ell(w) = 24$ and $\ell(\Phi(u)) = 21$. Therefore this is an embedding of $[u, v]$ into $[\Phi(u), w]$. $\square$

Example 2.3. Let $v = 2413 = s_1 s_3 s_2$, $u = 2143 = s_1 s_3$, and note that $\ell(v) - \ell(u) = 1$. Let $w = 265314$; note there are two embeddings Φ_1 and Φ_2 of v into w, represented respectively by the underlinings $\underline{265}3\underline{14}$ and $\underline{265}3\underline{14}$.

Neither of these embeddings induce an embedding of $[u, v]$. We have that $\Phi_1(u) = 215364$ and $\ell(w) - \ell(\Phi_1(u)) = 5 \neq \ell(v) - \ell(u) = 1$. (Note that $\ell(w) = 9$.) For Φ_2, $\Phi_2(u) = 261354$ and $\ell(w) - \ell(\Phi_2(u)) = 3$ which again differs from $\ell(v) - \ell(u) = 1$. Hence w in fact interval pattern avoids $[u, v]$, even though it does not classically pattern avoid v. $\square$

Two further lemmas follow immediately from our definition.

Lemma 2.4. *If Φ gives an embedding of $[u, v]$ into $[\Phi(u), w]$, then Φ also gives an embedding of $[u', v]$ into $[\Phi(u'), w]$ for any u' such that $u \leq u' \leq v$.*

Lemma 2.5. *If w avoids $[u, v]$, then w avoids $[u', v]$ for any $u' \leq u$.*

2.2. **Semicontinuously stable properties.** We are interested in local properties of Schubert varieties that are semicontinuous, meaning that they hold on closed subsets of any variety, and are preserved under products with affine space. We call such a property $\mathcal{P}$ **semicontinuously stable**. For example,

$$\{\text{semicontinuously stable } \mathcal{P}\} = \left\{ \begin{array}{c} \text{singular, non-Gorenstein,} \\ \text{non-factorial, dimension of i-th local intersection} \\ \text{cohomology group} \geq k, \text{Cohen-Macaulay type} \geq k, \\ \text{multiplicity} \geq k, \ldots \end{array} \right\}.$$

For us, "reasonable" invariants of singularities are properties such that they, or their negations, are semicontinuously stable. (Actually, at present there is no general result that the property $\mathcal{P} =$"dimension of i-th local intersection cohomology group $\geq k$" is semicontinuously stable, but for Schubert varieties this is known to be true [17].)

We desire a common combinatorial language to describe the $\mathcal{P}$**-locus**, the closed subset of a Schubert variety X_w at which the local property $\mathcal{P}$ holds. (As explained in Section 3 below, the $\mathcal{P}$-locus is a union of Schubert subvarieties, and it suffices to consider this question at the T-fixed points e_x for $x \leq w$.) As it turns out, classical pattern avoidance is insufficient in general for this purpose. We first observed this for Gorensteinness in [35]. (See Example 2.8 below.) It was also there that we first noticed the phenomenon of Bruhat-restricted/interval pattern avoidance, which suggested the present study.

To connect the combinatorics of interval pattern avoidance to the geometry of Schubert varieties, we need a little more notation and terminology. Consider the set

$$\mathfrak{S} = \{[u, v] : u \leq v \text{ in some } S_r\} \subseteq S_\infty \times S_\infty$$

where $S_\infty = \bigcup_{r \geq 1} S_r$.

Define $\prec$ to be the partial order on $\mathfrak{S}$ generated by the two types of relations

(1) $[u, v] \prec [x, w]$ if there is an interval pattern embedding of $[u, v]$ into $[x, w]$, and
(2) $[u, v] \prec [u', v]$ if $u' \leq u$.

An **upper order ideal** $\mathcal{I}$ (under the partial order $\prec$) is a subset of $\mathfrak{S}$ such that, if $[u, v] \in \mathcal{I}$ and $[u, v] \prec [x, w]$, then $[x, w] \in \mathcal{I}$.

We are now ready to state the precise version of our main idea from Section 1:

Theorem 2.6. *Let $\mathcal{P}$ be a semicontinuously stable property. The set of intervals $\{[u, v]\} \subseteq \mathfrak{S}$ such that $\mathcal{P}$ holds at the T-fixed point e_u on the Schubert variety X_v is an upper order ideal $\mathcal{I}_\mathcal{P}$ under $\prec$.*

We also wish to characterize Schubert varieties that globally avoid $\mathcal{P}$, or, in other words, those Schubert varieties for which $\mathcal{P}$ does not hold at any point, in analogy with the theorems for smoothness [23] and Gorensteinness [35]. The following corollary says that this can be done in terms of interval pattern avoidance.

Corollary 2.7. *Let $\mathcal{P}$ be a semicontinuously stable property. Then the set of permutations w such that $\mathcal{P}$ does not hold at any point of X_w is the set of permutations w that avoid all the intervals $[u_i, v_i]$ constituting some (possibly infinite) set $A_\mathcal{P} \subseteq \mathfrak{S}$.*

The corollary is false in general for classical pattern avoidance, as the following example illustrates:

Example 2.8. The Schubert variety $X_{42513} \subseteq \mathrm{Flags}(\mathbb{C}^5)$ is not Gorenstein (see Theorem 6.6). However $X_{526413} \subseteq \mathrm{Flags}(\mathbb{C}^6)$ is Gorenstein even though 42513 embeds into $\underline{526413}$ at the underlined positions. So Gorensteinness cannot be characterized using classical pattern avoidance. $\qquad\square$

We speculate that for any semicontinuously stable property $\mathcal{P}$, $\mathcal{I}_\mathcal{P}$ and $A_\mathcal{P}$ respectively provide natural answers to the problems of

- determining the $\mathcal{P}$-locus of X_w and
- characterizing which Schubert varieties X_w globally avoid $\mathcal{P}$.

Therefore, we expect interval pattern avoidance to be a useful framework for studying these questions, both in principle, as established by the above results, and practice, as evidenced by the examples in Section 6.

In Sections 3 and 4, we introduce Kazhdan-Lusztig ideals, using them to explain and prove Theorem 2.6 and its corollary. After this we will proceed to describe $\mathcal{I}_\mathcal{P}$ and $A_\mathcal{P}$ for the properties $\mathcal{P}$ for which they are known, and explain how $\mathcal{I}_\mathcal{P}$ and $A_\mathcal{P}$ might be computed for some other properties. Note that $\mathcal{I}_\mathcal{P}$ and $A_\mathcal{P}$ may vary depending on the base field $\Bbbk$ for some properties $\mathcal{P}$.

3. Schubert varieties and Kazhdan-Lusztig ideals

3.1. **Schubert definitions.** Let $G = \mathrm{GL}_n(\mathbb{C})$ denote the group of invertible $n \times n$ matrices with entries in $\mathbb{C}$, and let $B, B_-, T \subseteq G$ denote the subgroups of upper triangular, lower triangular and diagonal matrices respectively. The **complete flag variety** is

$\text{Flags}(\mathbb{C}^n) = G/B$; upon choosing a basis of $\mathbb{C}^n$, a point $gB \in G/B$ is naturally identified with a complete flag $F_\bullet : \langle 0 \rangle \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_n = \mathbb{C}^n$ by allowing F_i to be the span of the first i columns of any coset representative of gB.

The flag variety has a **Bruhat decomposition**

$$G/B = \coprod_{w \in S_n} BwB/B,$$

where we think of w as the permutation matrix with 1's at $(w(i), i)$ and 0's elsewhere. The Zariski closure of the **Schubert cell** $X_w^\circ := BwB/B$ is the Schubert variety $X_w := \overline{X_w^\circ}$. The Schubert cell X_w° is isomorphic to affine space $\mathbb{A}^{\ell(w)}$. Moreover, each Schubert variety is the disjoint union of Schubert cells

$$X_w = \coprod_{x \leq w} X_x^\circ.$$

Our conventions have been chosen so that the dimension of X_w is $\ell(w)$. In particular, X_{id} is a point, and $X_{w_0} = G/B$, where w_0 denotes the permutation such that $w_0(i) = n + 1 - i$.

The T-fixed points of X_w (under the left action of T on G/B) are $e_x := xB/B$ for $x \leq w$; these are known as **Schubert points** and represent the flags corresponding to permutation matrices. Every point on a Schubert variety is in the B-orbit of some Schubert point, and the B-action gives an isomorphism between a local neighborhood of any point with a local neighborhood of a Schubert point. Therefore, in studying local questions about singularities of Schubert varieties, we may restrict attention to the Schubert points.

3.2. **Affine neighborhoods, explicitly.** Let M_n be the space of $n \times n$ matrices over $\mathbb{C}$; we think of M_n as a variety with coordinate ring $\mathbb{C}[\mathbf{z}]$ where $\mathbf{z} := \{z_{n-i+1,j}\}_{i,j=1}^n$ are the entries of a generic matrix Z. (Note we are labeling our variables so that z_{11} is at the southwest corner (at row n and column 1) of the generic matrix.)

For a permutation $x \in S_n$, let $Z^{(x)}$ be the generic matrix obtained when we specialize Z by setting $z_{n-x(i)+1,i} = 1$ for all i, and $z_{n-x(i)+1,a} = 0$ and $z_{n-b+1,i} = 0$ for $a > i$ and $b < x(i)$. Let $\mathbf{z}^{(x)} \subseteq \mathbf{z}$ denote the other (unspecialized) variables. Let $Z_{ij}^{(x)}$ denote the southwest submatrix of $Z^{(x)}$ with northeast corner (i, j); this matrix has $n - i + 1$ rows and j columns. Furthermore, let $R^w = \left[r_{ij}^w\right]_{i,j=1}^n$ be the **rank matrix**, in which each r_{ij}^w equals the number of 1's to the southwest of (i, j) in w:

$$r_{ij}^w = \#\{k \mid w(k) \geq i \text{ and } k \leq j\}.$$

Let the **Kazhdan-Lusztig ideal** $I_{x,w}$ be generated by the size $1 + r_{ij}^w$ minors of $Z_{ij}^{(x)}$, over all possible i and j. Let

$$\mathcal{N}_{x,w} := \text{Spec}(\mathbb{C}[\mathbf{z}^{(x)}]/I_{x,w})$$

be the associated affine scheme. (See Example 3.4 below.)

We formulate our proof of Theorem 2.6 and the computations in Sections 5–6 using the following fact. (Experts will find the ideas contained herein familiar.)

Proposition 3.1. *$\mathcal{N}_{x,w} \times \mathbb{A}^{\ell(x)}$ is isomorphic to an affine neighborhood of X_w at e_x. In particular, if $\mathcal{P}$ is a semicontinuously stable property, then $\mathcal{P}$ holds at e_x on X_w if and only if $\mathcal{P}$ holds at the origin $\mathbf{0}$ on $\mathcal{N}_{x,w}$.*

Proof and discussion: An affine neighborhood of e_x in the flag variety is given by $x\Omega_{id}^\circ$, where, in general, Ω_u° is the **opposite Schubert cell** defined by $\Omega_u^\circ := B_-uB/B = w_0X_{w_0u}^\circ$. To study X_w locally at e_x, we therefore need only understand $X_w \cap x\Omega_{id}^\circ$.

We now proceed to describe explicit coordinates for $x\Omega_{id}^\circ$ and equations for $X_w \cap x\Omega_{id}^\circ$ in terms of these coordinates.

Let $\pi : G \to G/B$ be the natural quotient map. The map π has a local section σ over Ω_{id}° with $\sigma(F_\bullet)$ being the unique coset representative of $F_\bullet$ which is unit lower triangular (1's are on the main diagonal). The map $x\sigma x^{-1}$ is then a local section over $x\Omega_{id}^\circ$; therefore we have that

$$X_w \cap x\Omega_{id}^\circ \cong \pi^{-1}(X_w) \cap x\sigma x^{-1}(x\Omega_{id}^\circ),$$

where the latter can be considered as a subvariety of M_n.

The following lemma was first stated by D. Kazhdan and G. Lusztig (whence our terminology for $I_{x,w}$). It holds for the flag varieties of all algebraic groups. For completeness, we give an explicit description of the isomorphism in our $GL_n(\mathbb{C})$ case.

Lemma 3.2 (Lemma A.4 of [19]). $X_w \cap x\Omega_{id}^\circ \cong (X_w \cap \Omega_x^\circ) \times \mathbb{A}^{\ell(x)}$.

Proof. The map $x\sigma x^{-1}$ sends Ω_x° to the set of matrices with 1's at $(x(i), i)$ for $1 \le i \le n$, 0's to the right and above these 1's, and arbitrary entries elsewhere. Now identify $\mathbb{A}^{\ell(x)}$ with the space of unit upper triangular matrices $m = [m_{ij}]_{i,j=1}^n$ for which $m_{ij} = 0$ (for $i < j$) unless $x(i) > x(j)$. It is easy to check that the map $\eta : x\sigma x^{-1}(\Omega_x^\circ) \times \mathbb{A}^{\ell(x)} \to x\sigma x^{-1}(x\Omega_{id}^\circ)$ given by $\eta(a, m) = ma$ (where we have matrix multiplication on the right hand side) is an isomorphism. Now notice η restricts as desired to any Schubert variety X_w; if $a \in X_w \cap \Omega_x^\circ$, then $\pi(\eta(x\sigma x^{-1}(a), m)) \in X_w \cap x\Omega_{id}^\circ$, since X_w is closed under the action of B. $\qquad\square$

Let $\mathcal{N}'_{x,w}$ denote the variety $X_w \cap \Omega_x^\circ$. In view of Lemma 3.2, that it suffices to study these varieties to understand semicontinuously stable properties of X_w. We want to show $\mathcal{N}'_{x,w} \cong \mathcal{N}_{x,w}$.

To do this, we want explicit equations in coordinates for $\mathcal{N}'_{x,w}$. Since $x\sigma x^{-1} : \Omega_x^\circ \to G$ is a section of the map π and hence an injection, we have

$$\mathcal{N}'_{x,w} \cong x\sigma x^{-1}(\Omega_x^\circ) \cap \pi^{-1}(X_w).$$

One coordinate ring for $GL_n(\mathbb{C})$ is $\mathbb{C}[\mathbf{z}, \det^{-1}(\mathbf{z})]$, where $\mathbf{z} := (z_{n-i+1,j})_{i,j=1}^n$ are the entries of a generic *invertible* matrix Z. With these coordinates, the defining ideal for $x\sigma x^{-1}(\Omega_x^\circ)$ is generated by the polynomials $z_{n-x(i)+1,i} - 1$ and monomials of the form $z_{n-x(i)+1,a}$ and $z_{n-b+1,i}$ for $a > i$ and $b < x(i)$; we denote this ideal J_x. Fulton [13] showed that the ideal I_w defining $\pi^{-1}(X_w)$ (scheme-theoretically) is generated by the size $1 + r_{ij}^w$ minors of Z_{ij}, over all possible i and j; the closure of $\pi^{-1}(X_w)$ in M_n defined by $\mathbb{C}[\mathbf{z}]/I_w$ is known as the **matrix Schubert variety**. Actually, as Fulton [13] explains, I_w is generated by a much smaller subset of these minors, corresponding to the essential set conditions; we describe this in the example below.[2] Therefore,

$$\mathcal{N}'_{x,w} \cong \mathrm{Spec}(\mathbb{C}[\mathbf{z}]/(I_w + J_x)).$$

(Note that $\det(\mathbf{z}) = 1$ by J_x.)

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[2] Our conventions differ from those of [13, 20, 35]; our equations define their (matrix) Schubert varieties for w_0w^{-1}.

In practice, to reduce the number of variables, instead of working in a generic matrix Z, we first quotient by J_x and work in the generic matrix $Z^{(x)}$. The image of I_w in $\mathbb{C}[z^{(x)}]$ under this quotient by J_x is precisely $I_{x,w}$. Therefore $\mathcal{N}_{x,w} \cong \mathcal{N}'_{x,w}$ and the result follows. $\square$

The following is an immediate corollary. However, it is far from obvious if one looks only at the generators of $I_{x,w}$.

Corollary 3.3. $\mathcal{N}_{x,w}$ *is reduced and irreducible of dimension* $\ell(w) - \ell(x)$.

The following gives an example of the theorem as well as its proof and discussion above:

Example 3.4. Let $w = 35142 \in S_5$; then

$$
Z = \begin{pmatrix} z_{51} & z_{52} & z_{53} & z_{54} & z_{55} \\ z_{41} & z_{42} & z_{43} & z_{44} & z_{45} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} \\ z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \text{ and } R_w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 3 & 4 \\ 1 & 2 & 2 & 3 & 3 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}.
$$

Drawing "hooks" to the right and above every 1 in w defines the **diagram** of w, which is the set of positions not in any hook. Here, the diagram is the set $\{(2,3), (4,1), (4,3), (5,1)\}$. The **essential set** consists of the northeast most boxes in each connected component of the diagram, in this case, $(2,3)$, $(4,1)$ and $(4,3)$. Fulton [13] showed that the minors arising from considering just these three positions generate all of I_w, so

$$
I_w = \left\langle z_{11}, z_{21}, \begin{vmatrix} z_{31} & z_{32} & z_{33} \\ z_{21} & z_{22} & z_{23} \\ z_{11} & z_{12} & z_{13} \end{vmatrix}, \begin{vmatrix} z_{41} & z_{42} & z_{43} \\ z_{21} & z_{22} & z_{23} \\ z_{11} & z_{12} & z_{13} \end{vmatrix}, \begin{vmatrix} z_{41} & z_{42} & z_{43} \\ z_{31} & z_{32} & z_{33} \\ z_{11} & z_{12} & z_{13} \end{vmatrix}, \begin{vmatrix} z_{41} & z_{42} & z_{43} \\ z_{31} & z_{32} & z_{33} \\ z_{21} & z_{22} & z_{23} \end{vmatrix}, \right.
$$
$$
\left. \begin{vmatrix} z_{21} & z_{22} \\ z_{11} & z_{12} \end{vmatrix}, \begin{vmatrix} z_{21} & z_{23} \\ z_{11} & z_{13} \end{vmatrix}, \begin{vmatrix} z_{22} & z_{23} \\ z_{12} & z_{13} \end{vmatrix} \right\rangle.
$$

(The reader may find it helpful to argue why all the other minors in I_w are in the ideal generated by only these minors.)

Let $x = 13254 \le w$; then a generic matrix of $x\sigma x^{-1}(\Omega_x^\circ)$ is

$$
\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ z_{41} & 0 & 1 & 0 & 0 \\ z_{31} & 1 & 0 & 0 & 0 \\ z_{21} & z_{22} & z_{23} & 0 & 1 \\ z_{11} & z_{12} & z_{13} & 1 & 0 \end{pmatrix}.
$$

We set $z_{51} = z_{32} = z_{43} = z_{14} = z_{25} = 1$, and all other variables *except* $z_{11}, z_{12}, z_{13}, z_{21}, z_{22}, z_{23}, z_{31}, z_{41}$ to 0, resulting in the Kazhdan-Lusztig ideal

$$
\begin{aligned}
I_{x,w} = \; & \langle z_{11}, z_{21}, -z_{11}z_{23} + z_{21}z_{13} + z_{31}z_{12}z_{23} - z_{31}z_{13}z_{22}, z_{11}z_{22} - z_{21}z_{12} + z_{41}z_{12}z_{23} \\
& -z_{41}z_{13}z_{22}, z_{11} - z_{31}z_{12} - z_{41}z_{13}, z_{21} - z_{31}z_{22} - z_{41}z_{23}, z_{11}z_{22} - z_{21}z_{12}, \\
& z_{11}z_{23} - z_{21}z_{13}, z_{12}z_{23} - z_{22}z_{13}\rangle.
\end{aligned}
$$

$\square$

We remark that any matrix Schubert variety can be realized (up to a product with affine space) as a particular $\mathcal{N}_{x,w}$. Specifically, for $w \in S_n$, consider its natural embedding into

S_{2n} and let $x = \mathrm{id} \in S_{2n}$. Then it is not hard to check that the generators of $I_{\mathrm{id},w}$ are the same as the generators of I_w, except that the ideal I_w is defined in a ring with more variables.

4. A LOCAL ISOMORPHISM AND THE PROOF OF THEOREM 2.6, COROLLARY 2.7

The following lemma explains the role of semicontinuity in our discussion.

Lemma 4.1. *If $u' \leq u$ and X_v has a semicontinuously stable property $\mathcal{P}$ at e_u, then X_v must also have this property at $e_{u'}$. In particular, if property $\mathcal{P}$ fails to hold at e_{id}, it will not hold for any point on X_v.*

Proof. It suffices to consider the case where u actually covers u' in Bruhat order. There is a $\mathbb{P}^1$ connecting $e_{u'}$ and e_u. In this $\mathbb{P}^1$, the generic point is in the B-orbit of e_u, and hence has a neighborhood isomorphic to a neighborhood of e_u. The one special point is e'_u. Since $\mathcal{P}$ is semicontinuous and holds on a generic point of this $\mathbb{P}^1$, it must hold at $e_{u'}$.

The second assertion is a special case of the contrapositive of the first. □

Our proof of Theorem 2.6 follows from the following local isomorphism result, which explains the geometric significance of interval pattern embeddings.

Theorem 4.2. *Let Φ be an interval pattern embedding of $[u,v]$ into $[x,w]$. Then there exists a scheme-theoretic isomorphism $\mathcal{N}_{u,v} \cong \mathcal{N}_{x,w}$. Therefore, affine neighborhoods of X_v and X_w respectively at e_u and e_x are isomorphic up to cartesian products with affine space.*

Note that if Φ does not induce an interval isomorphism of $[u,v]$ and $[x,w]$, then $\mathcal{N}_{u,v}$ and $\mathcal{N}_{x,w}$ are not isomorphic since their dimensions differ.

See Example 4.5 below for an illustration of the arguments in the following proof.

Proof of Theorem 4.2: Let $I = \{i_1 < \ldots < i_m\}$ be the embedding indices of Φ and let $\{1, 2, \ldots, n\} \setminus I = \{j_1 < \ldots < j_{n-m}\}$.

Lemma 4.3. *For $1 \leq d \leq n - m$,*

(1) $$\#\{k : k \leq j_d \text{ and } \Phi(u)(k) \geq \Phi(u)(j_d)\} = \#\{k : k \leq j_d \text{ and } w(k) \geq w(j_d)\}.$$

Proof. Let $s_{\alpha_1} \cdots s_{\alpha_{\ell(v)-\ell(u)}}$ be a reduced expression of $v^{-1}u$ and consider the corresponding product of (possibly non-simple) transpositions $t_1 \cdots t_{\ell(v)-\ell(u)}$ where $t_j = (i_{\alpha_j} \leftrightarrow i_{\alpha_j+1})$.

Note that since Φ is an interval pattern embedding of $[u,v]$ into $[x,w]$, as we successively multiply w on the right by the transpositions $t_1, t_2, \ldots, t_{\ell(v)-\ell(u)}$, each transposition drops the Coxeter length by exactly 1 (because each decreases the Coxeter length, and the total drop in length is $\ell(v) - \ell(u)$). Also observe that for any permutation p and transposition $t = (a \leftrightarrow b)$, $\ell(pt) = \ell(p) - 1$ if and only if $p(a) > p(b)$ and there does not exist an index k, $a < k < b$, such that $p(a) > p(k) > p(b)$.

For each c, $0 \leq c \leq \ell(v) - \ell(u)$, define the permutation $w^{(c)}$ by $w^{(c)} := wt_1 \cdots t_c$. We can now check that each $w^{(c)}$ satisfies

$$\#\{k : k \leq j_d \text{ and } w^{(c)}(k) \geq w^{(c)}(j_d)\} = \#\{k : k \leq j_d \text{ and } w^{(c-1)}(k) \geq w^{(c-1)}(j_d)\}$$

for all d. If this equation were to fail for any d, it would have to be the case that $t_c = (a \leftrightarrow b)$ with $a < j_d < b$ and $w^{(c-1)}(a) > w^{(c-1)}(j_d) > w^{(c-1)}(b)$.

Since $w^{(\ell(v)-\ell(u))} = \Phi(u)$, the lemma follows by induction. □

The following lemma shows the vanishing of certain coordinates at all points of $N_{\Phi(u),w}$.

Lemma 4.4. *Let* $g = (g_{ij}) \in \mathcal{N}_{\Phi(u),w}$. *Then for each* $1 \le d \le n - m$ *we have* $g_{\Phi(u)(j_d),j_d} = 1$, $g_{a,j_d} = 0$ *for any* $a \ne \Phi(u)(j_d)$ *and* $g_{\Phi(u)(j_d),b} = 0$ *for any* $b \ne j_d$.

Proof. Since $J_{\Phi(u)}$ vanishes on g, $g_{\Phi(u)(j_d),j_d} = 1$, $g_{a,j_d} = 0$ for $a < \Phi(u)(j_d)$ and $g_{\Phi(u)(j_d),b} = 0$ for $b > j_d$. It remains to check the cases $a > \Phi(u)(j_d)$ and $b < j_d$.

Now we check the case $b < j_d$. Since $w(j_d) = \Phi(u)(j_d)$, both have a "1" in position $(w(j_d), j_d) = (\Phi(u)(j_d), j_d)$. Moreover, by Lemma 4.3, $r^w_{w(j_d),j_d} = r^{\Phi(u)}_{\Phi(u)(j_d),j_d}$. Let this common integer be S. Then $r^w_{w(j_d),j_d-1} = S - 1$ since the "1" in position $(w(j_d), j_d)$ causes the rank to increase by 1 as one moves from $(w(j_d), j_d - 1)$ to $(w(j_d), j_d)$. Hence the $S \times S$ minors of the southwest $(n - w(j_d) + 1) \times (j_d - 1)$ submatrix of g vanishes. Furthermore, $r^{\Phi(u)}_{\Phi(u)(j_d),j_d-1} = S - 1$, so there are $S - 1$ rows in this submatrix with an entry of "1" in their rightmost nonzero columns; these rightmost columns are distinct from each other. The generic matrix $Z^{(\Phi(u))}$ has a "0" in row $\Phi(u)(j_d)$ in these columns; if some other entry in row $\Phi(u)(j_d)$ has a nonzero entry, then the submatrix would have S linearly independent columns, a contradiction. Therefore, $g_{\Phi(u)(j_d),b} = 0$ for any $b < j_d$.

The proof for the case $a > \Phi(u)(j_d)$ is similar. $\square$

Define the (algebraic) map $\Psi : \mathcal{N}_{\Phi(u),w} \to u\sigma u^{-1}(\Omega^\circ_u)$ as the projection which deletes the columns $j_1, \ldots, j_{n-m}$ and rows $w(j_1), \ldots, w(j_{n-m})$ from an element $g \in \mathcal{N}_{\Phi(u),w}$. This map is well defined, and, by Lemma 4.4, injective.

Next, we show that the image of Ψ is actually inside $\mathcal{N}_{u,v}$. This amounts to verifying that $\Psi(g)$ satisfies the southwest rank conditions corresponding to the minors generating $I_{u,v}$. Consider the southwest submatrix of $\Psi(g)$ with northeast corner (a, b); let $b' = i_b$ and $a' = \Phi(u)(i_{u^{-1}(a)})$ be the corresponding indices for g. Observe that by the definition of the rank matrix, $r^{\Phi(u)}_{a'b'} = r^u_{ab} + c_{ab}$ where c_{ab} equals the number of positions of the form $(w(j_d), j_d)$ southwest of (a', b'), which by Lemma 4.3 is the number of positions of the form $(\Phi(u)(j_d), j_d)$ weakly southwest of (a', b'). But by Lemma 4.4, removing row $\Phi(u)(j_d)$ and column j_d must drop the rank by exactly 1, since those rows and columns have a "1" at $(\Phi(u)(j_d), j_d)$ and "0" everywhere else. Therefore, deleting the aforementioned rows and columns precisely drops the rank at (a', b') by precisely c_{ab}, so $\Psi(g)$ satisfies exactly the rank conditions for $\mathcal{N}_{u,v}$. Therefore the image of Ψ is inside $\mathcal{N}_{u,v}$.

On the other hand, given a point in $\mathcal{N}_{u,v}$ we can add back these deleted rows and columns. This is clearly the inverse map to Ψ, so it follows that Ψ is an isomorphism from $\mathcal{N}_{\Phi(u),w}$ to $\mathcal{N}_{u,v}$. $\square$

We illustrate this theorem (and its proof) by the following example.

Example 4.5. Let $v = 35142, u = 13524$ and $w = \underline{5}89\underline{7}16234$ be as in Example 3.4, with Φ indicated by the underlined positions, so $x = \Phi(u) == 189573264$. The intervals $[u, v]$ and $[x, w]$ are easily checked to be isomorphic. A generic matrix of $u\sigma u^{-1}(\Omega^\circ_u)$ has the form

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
z_{41} & 0 & 0 & 1 & 0 \\
z_{31} & 1 & 0 & 0 & 0 \\
z_{21} & z_{22} & 0 & z_{24} & 1 \\
z_{11} & z_{12} & 1 & 0 & 0
\end{pmatrix}, \text{ while } v =
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix}, \text{ and } R^v =
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
1 & 2 & 2 & 3 & 4 \\
1 & 2 & 2 & 3 & 3 \\
0 & 1 & 1 & 2 & 2 \\
0 & 1 & 1 & 1 & 1
\end{pmatrix}.
$$

The reader can check that:

$$\mathcal{N}_{u,v} \cong \mathrm{Spec}\left(\frac{\mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}, z_{24}, z_{31}, z_{41}]}{\langle z_{11}, z_{21}, z_{22}, z_{41}\rangle}\right) \cong \mathbb{A}^3.$$

Now,

$$w = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } R^w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 4 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 4 & 5 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 4 & 5 & 5 & 5 & 6 \\ 1 & 2 & 3 & 4 & 4 & 5 & 5 & 5 & 5 \\ 0 & 1 & 2 & 3 & 3 & 4 & 4 & 4 & 4 \\ 0 & 1 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

A generic matrix in $x\sigma x^{-1}(\Omega_x^\circ)$ has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_{81} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ y_{71} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ y_{61} & 0 & 0 & 0 & 0 & y_{66} & y_{67} & 0 & 1 \\ y_{51} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ y_{41} & 0 & 0 & y_{44} & 0 & y_{46} & y_{47} & 1 & 0 \\ y_{31} & 0 & 0 & y_{34} & 1 & 0 & 0 & 0 & 0 \\ y_{21} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_{11} & y_{12} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \text{ Then } \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ y_{71} & 0 & 0 & 1 & 0 \\ y_{51} & 1 & 0 & 0 & 0 \\ y_{41} & y_{44} & 0 & y_{46} & 1 \\ y_{31} & y_{34} & 1 & 0 & 0 \end{pmatrix}$$

is the result of applying Ψ to this generic matrix, since Ψ removes columns $2, 3, 7,$ and 9 and rows $2, 4, 8,$ and 9. The map Ψ is injective since, by Lemma 4.4, $y_{11}, y_{12}, y_{21}, y_{47}, y_{61}, y_{66}, y_{67},$ and y_{81} all equal 0 for any point in $\mathcal{N}_{x,w}$.

Now examine the rank conditions imposed by I_w, which can be read from the rank matrix R^w. We have that $y_{31} = y_{41} = 0$. Furthermore, since $r_{6,5}^w = 3$, all 4×4 minors of the southwest 4×5 submatrix (of an element of $\mathcal{N}_{x,w}$) vanish. It follows that $y_{44} = y_{41} = 0$. The remaining rank conditions imposing no further equations, it follows that $\mathcal{N}_{x,w} \cong \mathbb{A}^3$, in agreement with Theorem 2.6. $\qquad\square$

Conclusion of the proofs of Theorem 2.6 and Corollary 2.7: Let $[u, v] \in \mathfrak{S}$ be as in the statement of the theorem. Suppose $[u, v] \prec [x, w]$. We may assume that this is a covering relation. There are two cases. If Φ is an embedding of v into w and $x = \Phi(u)$ then the result follows from Theorem 4.2. In the other case, $[x, w] = [u', v]$ where $u' \le u$, and the conclusion holds by Lemma 4.1.

The corollary is the contrapositive of the theorem, with $A_\mathcal{P}$ being a generating set for $\mathcal{I}_\mathcal{P}$. $\qquad\square$

5. COMPUTING WITH KAZHDAN-LUSZTIG IDEALS

Next we turn to computing properties of Kazhdan-Lusztig ideals $I_{u,v}$ using Macaulay 2. Various computations have been automated in our scripts Schubsingular. We illustrate the main computations through specific examples here. Our computations are over $\mathbb{Q}$

and valid for any field of characteristic 0; similar computations can be made in other characteristics.

These computations can be used to conjecture generators $A_\mathcal{P}$ for the order ideal $\mathcal{I}_\mathcal{P}$ for various particular properties $\mathcal{P}$.

To begin explaining the computations, let us re-examine Example 3.4, in which $u = 13254$ and $v = 35142$.

```
R = QQ[z11, z12, z13, z14, z15,
       z21, z22, z23, z24, z25,
       z31, z32, z33, z34, z35,
       z41, z42, z43, z44, z45,
       z51, z52, z53, z54, z55]; n=5;  -- graded diagonal term order on G
                                       -- below
G = matrix({{  1,   0,   0,   0, 0},
            {z41,   0,   1,   0, 0},
            {z31,   1,   0,   0, 0},
            {z21, z22, z23,   0, 1},
            {z11, z12, z13,   1, 0}});  -- generic matrix in opposite
                                        -- Schubert cell of u
Rank = matrix({{1, 2, 3, 4 ,5},
               {1, 2, 2, 3, 4},
               {1, 2, 2, 3, 3},
               {0, 1, 1, 2, 2},
               {0, 1, 1, 1, 1}});       -- rank matrix of v
Jlist = trim(sum(flatten(for i from 0 to n-1 list
                        for j from 0 to n-1 list minors(Rank_(n-i-1,j)+1,
                        submatrix(G, {(n-i-1)..n-1}, {0..j})))))
```

The last line computes the generators for $I_{u,v}$. $\qquad\qquad\qquad\qquad\qquad\square$

The following problem is the next step in all of our further calculations, so an explicit combinatorial answer would speed up these computations.

Problem 5.1. *Find a Gröbner basis for* $I_{u,v}$.

It seems plausible to us that the defining determinants of $I_{u,v}$ are a Gröbner basis with respect to any graded diagonal term order in any characteristic. (This has been verified computationally for $n = 4, 5$, with respect to the canonical graded diagonal term order as used above). After re-writing our matrices upside down, this conjecture would generalize the conclusion for matrix Schubert varieties found in [20]. (In the process of preparing this report, we mentioned this possibility to A. Knutson, who informed us that he had independently discovered this generalization.)

Finding a Gröbner basis is the first step towards computing a free resolution of $I_{u,v}$. We point out that explicitly finding a (minimal) free resolution for even the special case of matrix Schubert varieties is an open problem of considerable interest that has been solved only in certain special cases. We believe that free resolutions for Kazhdan-Lusztig ideals in general, or even further special cases, are also of interest.

When an ideal of a polynomial ring is homogeneous under some positive grading, there is a unique minimal free resolution which is a subcomplex of every free resolution.

(A positive grading is one for which the degree 0 piece is $\mathbb{C}$ and the remainder of the ring has positive degree.) Unfortunately, $I_{u,v}$ is not in general a homogeneous ideal under the naive grading with $\deg z_{ij} = 1$ for all i, j. However, there is a positive grading of $\mathbb{C}[\mathbf{z}^{(u)}]$ under which $I_{u,v}$ is homogeneous.

Lemma 5.2. *Let $e_1, \ldots, e_n$ be generators of the group $\mathbb{Z}^n$. Under the multi-grading where the variable $z_{n-u(i)+1,j}$ has degree $e_i - e_j$, every variable in $\mathbb{C}[\mathbf{z}^{(u)}]$ has a degree which is a positive sum of the degrees $e_{i+1} - e_i$ for $1 \leq i \leq n-1$, and $I_{u,v}$ is homogeneous.*

Proof. We can assign the degree $e_{u^{-1}(i)} - e_j$ to the entry in row i and column j of the generic matrix $Z^{(u)}$ to get this grading, since a 0 can be assigned any degree, the 1's, which are at $(u(j), j)$ for $1 \leq j \leq n$, are assigned the degrees $e_j - e_j = 0$, and the variables z_{ij} are assigned the degrees specified. Therefore, the minor of $Z^{(u)}$ using rows $i_1, \ldots, i_k$ and columns $j_1, \ldots, j_k$ will be homogeneous of degree $\sum_{m=1}^{k} e_{u^{-1}(i_m)} - \sum_{m=1}^{k} e_{j_m}$. Since $I_{u,v}$ is generated by minors of $Z^{(u)}$, this proves $I_{u,v}$ is homogeneous under this grading.

For positivity, note that $Z^{(u)}$ has a variable $z_{n-u(i)+1,j}$ at the position $(u(i), j)$ only if $j < i$. $\qquad\square$

If we wish to have a $\mathbb{Z}$-grading, we can coarsen the above grading by sending $e_{i+1} - e_i$ to 1, which sends $e_i - e_j$ to $i - j$; this coarser grading is also positive. Also, note that our isomorphism Ψ of Theorem 4.2 is compatible with the map of multigradings sending e_{ϕ_t} to e_t. (Our multigrading secretly comes from the action of T on G/B.)

Problem 5.3. *Determine a (minimal) free resolution of $I_{u,v}$.*

The Betti number $\beta_i(I_{u,v})$ is the rank of the free module F_i in a minimal free resolution

$$0 \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots \leftarrow F_i \leftarrow \ldots \leftarrow 0.$$

If we also keep track of the (multi-)degrees of the generators of F_i, we get (multi-)graded Betti numbers. One problem which may be an easier step towards finding a free resolution is the following.

Problem 5.4. *Give a combinatorial method for finding the (multi-graded) betti numbers of $I_{u,v}$.*

It would then be left only to determine the maps. Note that answers to both of these problems are known to depend on the characteristic of the field.

Continuing further with our re-examination of Example 3.4, let us compute a minimal free resolution with respect to the aforementioned grading on $I_{u,v}$.

```
S = QQ[z11, z12, z13,
       z21, z22, z23,
       z31,
       z41, Degrees=>{{3},{2},{1},
                      {4},{3},{2},
                      {1},
                      {2}}];  -- S homogeneous with respect to
                              -- the nonstandard grading

Jlist = substitute(Jlist, S)  -- convert Jlist into an ideal of S
```

```
Resl  = res(S^1/Jlist) -- free resolution as an S-module

        1       5       9       7       2
o2 = S   <-- S   <-- S   <-- S   <-- S   <-- 0

        0       1       2       3       4       5



o2 : ChainComplex

Resl.dd_1 -- minimal generators of Jlist

o3 = | z11 z12z31+z13z41 z21 z13z22-z12z23 z22z31+z23z41 |

              1        5
o3 : Matrix S   <--- S

Resl.dd_2 -- first syzygies, etc

o4 = {3} | 0     -z12z31-z13z41 0      -z21 -z13z22+z12z23 0
     {3} | -z23 z11              -z22 0     0               -z21
     {4} | 0     0               0     z11  0               z12z31+z13z41
     {4} | -z31 0                z41  0     z11             0
     {4} | z13  0                z12  0     0               0

                            -z22z31-z23z41 0                0              |
                            0              0                0              |
                            0              -z13z22+z12z23   -z22z31-z23z41 |
                            0              z21              0              |
                            z11            0                z21            |

             5        9
o4 : Matrix S   <--- S

betti(Resl) -- degrees are for the grading mentioned above

o5 = total: 1 5 9 7 2
        0: 1 . . . .
        1: . . . . .
        2: . 2 . . .
        3: . 3 1 . .
        4: . . 2 . .
        5: . . 4 1 .
        6: . . 2 2 .
        7: . . . 2 .
        8: . . . 2 1
        9: . . . . 1
```

(In `Schubsingular`, this resolution is obtained using `minresKL({2,4,0,3,1},{0,2,1,4,3})`. Take note of the "computer indexing" which shifts the numbers down by 1.)

Although the above grading is natural in some ways, it also causes some problems as our ring $\mathbb{C}[z^{(u)}]$ is not generated in degree 1. Therefore, we are also interested in the following problem.

Problem 5.5. *Find an explicit characterization of the pairs* $(u, v) \in S_n \times S_n$ *for which* $I_{u,v}$ *is homogeneous under the standard grading* $\deg z_{ij} = 1$.

An obvious subset of such pairs consists of those where the essential set of v is contained inside the "staircase" defined by the 1s of u.

6. CALCULATIONS FOR SINGULARITY INVARIANTS

In this section, we discuss various semicontinuously stable properties $\mathcal{P}$. In many instances, we present or conjecture a nonrecursive combinatorial description of $\mathcal{I}_{\mathcal{P}}$ or $A_{\mathcal{P}}$. In other instances, we explain computational (`Macaulay 2`) aspects of the effort to find them.

6.1. **Smoothness.** The problem of determining the singular locus of X_w was solved independently by [3, 11, 18, 26]; see also the earlier work [14]. We can restate this problem in terms of interval pattern avoidance as that of finding a full set of generators for the ideal $\mathcal{I}_{\text{singular}}$. It is not difficult to verify that the following is a restatement of the singular locus theorem. (In what follows, we use the convention that the segment "$j \cdots i$" means $j, j-1, j-2, \ldots, i+1, i$. In particular, if $j < i$ then the segment is empty.)

Theorem 6.1. *The order ideal* $\mathcal{I}_{\text{singular}}$ *is minimally generated by the following families of intervals:*

  (1) $\big[(a+1)a\cdots 1(a+b+2)\cdots (a+2), \quad (a+b+2)(a+1)a\cdots 2(a+b+1)\cdots (a+2)1\big]$
      for all integers $a, b > 0$.
  (2) $\big[(a+1)\cdots 1(a+3)(a+2)(a+b+4)\cdots (a+4), \quad (a+3)(a+1)\cdots 2(a+b+4)1(a+b+3)\cdots (a+4)(a+2)\big]$ *for all integers* $a, b \geq 0$.
  (3) $\big[1(a+3)\cdots 2(a+4), \quad (a+3)(a+4)(a+2)\cdots 312\big]$ *for all integers* $a > 1$.

Example 6.2. We compute the singular locus of X_{461253} by the above theorem. After calculating the pairs that arise for S_n, $n \leq 6$, we have only an embedding of $[1324, 3412]$ into the first four positions of 461253, and two embeddings of $[13254, 35142]$, one excluding position 3 and the other excluding position 4. Therefore, X_{461253} is singular at $e_{u'}$ if and only if $u' \leq 142653$, $u' \leq 241365$, or $u' \leq 143265$ in Bruhat order. In other words, the singular locus decomposes as

$$\text{sing}(X_{461253}) = X^\circ_{142653} \cup X^\circ_{241365} \cup X^\circ_{143265}.$$

$\square$

The property $\mathcal{P} = $"singular" has the special feature that the set of permutations appearing as the top element of intervals in $\mathcal{I}_{\text{singular}}$ is the order ideal generated by 4231 and 3412 in the partial order given by *classical* pattern avoidance, where "u is smaller than v" if u classically embeds into v. This is the theorem of Lakshmibai and Sandhya [23] stated in Section 1.

This special feature does not hold in general for all semicontinuously stable properties. (Compare Conjecture 6.7 below with Example 2.8). On the other hand, Billey and Braden [1] have given a geometric explanation of why ordinary pattern avoidance characterizes smoothness. One would like to know which other semicontinuously stable properties $\mathcal{P}$ ordinary pattern avoidance characterizes. We also wonder if some feature of the combinatorics of $\mathcal{I}_\mathcal{P}$ might characterize when ordinary pattern avoidance actually suffices.

6.2. **Kazhdan-Lusztig polynomials.** Associated to each pair of permutations $v, w \in S_n$ with $v \leq w$ is the **Kazhdan-Lusztig polynomial** $P_{v,w}(q) \in \mathbb{N}[q]$. Although these polynomials have an elementary recursive definition [19], their combinatorics is difficult to understand. This is one motivation for studying the singularities of Schubert varieties.

Geometrically, $P_{v,w}(q)$ is the Poincaré polynomial for the local intersection cohomology of X_w at e_v. (Given this is a topological invariant, the problem of calculating Kazhdan-Lusztig polynomials only makes sense over the field $\mathbb{C}$.) In particular (in type A), $P_{v,w}(q) = 1$ if and only if X_w is smooth at e_v. Although dimensions of local intersection cohomology groups are not in general (upper or lower) semicontinuous, a result of Irving [17] (see also [5]) shows that they behave in an upper semicontinuous manner on Schubert varieties. Therefore, we can study Kazhdan-Lusztig polynomials using interval pattern avoidance.

It is an longstanding, well-known open problem to find (hopefully nonrecursive) manifestly positive, combinatorial rules for $P_{v,w}(q)$. Such rules are only known in a limited cases, essentially those where a semi-small resolution of singularities is known [4, 25]. The following corollary of Theorem 4.2 appears to be new. It generalizes a lemma of P. Polo [28, Lemma 2.6] (see also [1, Theorem 6]) which states the case where the embedding Φ is in consecutive positions or the entries in the positions of Φ are numerically consecutive.

Corollary 6.3. *Suppose* $[u, v]$ *and* $[x, w]$ *are isomorphic because of an interval pattern embedding. Then* $P_{x,w}(q) = P_{u,v}(q)$.

Lusztig's **interval conjecture** asserts that $P_{a,b}(q) = P_{v,w}(q)$ whenever the Bruhat order intervals $[a, b]$ and $[v, w]$ are isomorphic as posets; this conjecture is discussed with further references in [6, 7]. Corollary 6.3 confirms a new (albeit, very special) case of the conjecture.

Example 6.4. Ordinary pattern avoidance does not suffice for Corollary 6.3. Consider $v = 4231$ embedding into $\underline{52}3\underline{41}$ at the indicated positions. Let $u = 2143 \leq v$. So $\Phi(u) = 21354$. Then $P_{u,v} = 1 + q$ while $P_{\Phi(u),w} = 1 + 2q + q^2$. (Here we have made use of Goresky's tables for Kazhdan-Lusztig polynomials [15].)

At present, we have no counterexample to the analogue of the interval conjecture for any of the semicontinuously stable properties studied in this paper. Therefore, the much stronger assertion that $\mathcal{N}_{a,b} \cong \mathcal{N}_{u,v}$ whenever $[a, b]$ and $[u, v]$ are isomorphic as intervals in Bruhat order remains possible, though in our opinion extremely unlikely.

As with all of the numerical invariants studied here, understanding of where they increase is a basic issue of interest. This suggests a new incremental formulation of the aforementioned Kazhdan-Lusztig polynomial problem.

Problem 6.5. *Let $\mathcal{P}_{k,\ell}$ to be the property "the coefficient of q^ℓ in $P_{u,v}(q)$ is at least k" (or equivalently "$\dim_{\mathbb{C}} \mathrm{IH}^\ell_{e_u}(X_v) \geq k$"). Determine $\mathcal{I}_{\mathcal{P}_{k,\ell}}$ for various values of k and ℓ.*

As with all such problems in this paper, as a first step we can formulate the computational challenge of analyzing $\mathcal{I}_{\mathcal{P}}^{(n)}$ for small n, where $\mathcal{I}_{\mathcal{P}}^{(n)}$ is the set of intervals from $\mathcal{I}_{\mathcal{P}}$ from S_m for $m \leq n$.

Unfortunately, there appears to be no known applicable method for computing the ranks of local intersection cohomology groups directly from the equations defining a variety.

6.3. **The Gorenstein property and Cohen-Macaulay type.** A local ring $(R, \mathfrak{m}, \mathbb{k})$ is said to be **Cohen-Macaulay** if $\mathrm{Ext}^i_R(\mathbb{k}, R) = 0$ for $i \leq \dim R$; it is **Gorenstein** if, in addition, $\dim_{\mathbb{k}} \mathrm{Ext}^{\dim R}_R(\mathbb{k}, R) = 1$. A variety is Cohen-Macaulay (respectively Gorenstein) if the local ring at every point is Cohen-Macaulay (respectively Gorenstein). Using the Kozsul complex on a regular sequence, one can show that every regular local ring is Gorenstein; hence smooth varieties are Gorenstein. See [10] for details.

In [35] we characterized the Schubert varieties which are Gorenstein at all points. Here is a reformulation of the main result of that paper purely in terms of interval pattern avoidance.

Theorem 6.6. *The Schubert variety X_w is Gorenstein if and only if w avoids the following intervals*

(1) $\big[(a+1)a\cdots1(a+b+2)\cdots(a+2), \quad (a+b+2)(a+1)a\cdots2(a+b+1)\cdots(a+2)1\big]$ *for all integers $a, b > 0$ such that $a \neq b$.*

(2) $\big[(a+1)\cdots1(a+3)(a+2)(a+b+4)\cdots(a+4), \quad (a+3)(a+1)\cdots2(a+b+4)1(a+b+3)\cdots(a+4)(a+2)\big]$ *for all integers $a, b \geq 0$, with either $a > 0$ or $b > 0$.*

As is explained in [35], the above theorem shows that a Schubert variety is Gorenstein if and only if the generic points of its singular locus are. This is closely related to the following conjecture, which we now restate in the terminology of interval pattern avoidance.

Conjecture 6.7. *The order ideal $\mathcal{I}_{\text{not Gorenstein}}$ is generated by the following families:*

(1) $\big[(a+1)a\cdots1(a+b+2)\cdots(a+2), \quad (a+b+2)(a+1)a\cdots2(a+b+1)\cdots(a+2)1\big]$ *for all integers $a, b > 0$ such that $a \neq b$.*

(2) $\big[(a+1)\cdots1(a+3)(a+2)(a+b+4)\cdots(a+4), \quad (a+3)(a+1)\cdots2(a+b+4)1(a+b+3)\cdots(a+4)(a+2)\big]$ *for all integers $a, b \geq 0$, with either $a > 0$ or $b > 0$.*

The components of the singular locus whose generic points are not Gorenstein are clearly in the non-Gorenstein locus. *A priori*, it is possible for some Schubert variety to be non-Gorenstein at some non-generic points outside of these components known to be non-Gorenstein. The content behind the conjecture is that this does not occur. Translating this geometric assertion into the combinatorics of interval pattern avoidance, we get the conjecture that the generators of $\mathcal{I}_{\text{not Gorenstein}}$ is the subset of the minimal generators (u, v) of $\mathcal{I}_{\text{singular}}$ for which X_v is not Gorenstein at e_u. Using the description of neighborhoods of generic points of the singular locus given in [11, 27], we arrived at the above conjecture, which has been partially verified by computations as described below.

The **Cohen-Macaulay type** of a local Cohen-Macaulay ring is defined to be $\dim_{\Bbbk} \mathrm{Ext}_R^{\dim R}(\Bbbk, R)$. This is a numerical refinement of the binary question of whether a variety is Gorenstein at a point. In our case, where the ring $R = \mathbb{C}[\mathbf{z}^{(u)}]/I_{u,v}$ is given as a quotient of a polynomial ring $S = \mathbb{C}[\mathbf{z}^{(u)}]$, Cohen-Macaulay type can be calculated as the last non-zero Betti number by the following well-known argument. We recall it here for completeness, as we could not find an explicit reference.

We have that $\mathrm{Ext}_R^{\dim R}(\Bbbk, R) = \mathrm{Ext}_S^{\dim R}(\Bbbk, R)$. Now let $K_\bullet$ be the Kozsul complex which is a free resolution of $\Bbbk$ (over S); we then can calculate the Ext module using the definition $\mathrm{Ext}_S^{\dim R}(\Bbbk, R) = H^{\dim R}(\mathrm{Hom}_S(K_\bullet, R))$. Since $K_\bullet$ is self-dual of length $\dim S$, we have that $H^{\dim R}(\mathrm{Hom}_S(K_\bullet, R)) = H_{\dim S - \dim R}(K_\bullet \otimes R) = \mathrm{Tor}_{\mathrm{codim}\,R}^S(\Bbbk, R)$. Now we can calculate $\mathrm{Tor}_{\mathrm{codim}\,R}^S(\Bbbk, R)$ using a free resolution of R as an S-module, and the Cohen-Macaulay type can be calculated as the last (since Schubert varieties are Cohen-Macaulay) non-zero Betti number of R as an S-module.

Computation gives us the following partial check of our conjecture. Note the computations have only been done in characteristic 0.

Proposition 6.8. *Conjecture 6.7 is true for* $n \le 6$.

Proof. The conjecture is vacuously true for $n \le 4$ and for $n = 5$ the result follows from [35, Corollary 1], since every non-Gorenstein Schubert variety for $n = 5$ has only one component in its singular locus, so every singular point is non-Gorenstein. The verification for $n = 6$ is by computer. $\square$

The computation verifying the conjecture for $n = 6$ is already somewhat involved, as the following example shows.

Example 6.9. Let $v = 461253$ as in the previous example. The conjecture states that X_{461253} is not Gorenstein at $e_{u'}$ if and only if $u' \le 241365$ or $u' \le 143265$ in Bruhat order. There are 24 elements of S_n in the interval between v and id (inclusive). Of those, 9 are in fact smaller that 241365 or 143265 in the Bruhat order, namely:

$$142365, 124365, 132465, 142356, 123465, 124356, 132456, \mathrm{id}.$$

All of the remainder are larger than $u = 123546$ in the Bruhat order. Therefore, one only needs to compute a free resolution for $\mathcal{N}_{u,v}$. `Macaulay 2` reveals the following.

```
        1       7      21       35       35       21       7       1
o4 = S   <-- S  <-- S    <-- S    <-- S    <-- S    <-- S  <-- S  <-- 0

        0       1       2        3        4        5       6       7       8
```

o4 : ChainComplex

The last nonzero free module in the resolution is rank 1, agreeing with the conjecture. (Using `Schubsingular` the function `locuscompute` automates the Bruhat order analysis needed in the above computation.) $\square$

The most general problem for this invariant is to calculate the Cohen-Macaulay type of X_v at e_u. As with the problem of calculating Kazhdan-Lusztig polynomials, this problem can also be reformulated in an incremental form as follows.

Problem 6.10. *Let $\mathcal{P}_k$ be the property "canonical sheaf of X_v has rank at least k". Find the generators for the ideal $\mathcal{I}_{\mathcal{P}_k}$ for all k.*

It may be particularly interesting to understand what changes in the singularity structure cause the Cohen-Macaulay type to change, leading to the following question.

Problem 6.11. *Characterize pairs (u, v) such that the Cohen-Macaulay type of X_v at e_u is larger than the Cohen-Macaulay type X_v at $e_{u'}$ for all u' with $u < u' \leq v$.*

One important value for u is the identity permutation, since, by semicontinuity, the Cohen-Macaulay type of X_v will be largest at that point. Therefore, the following case of Problem 6.10 is of particular interest.

Problem 6.12. *Find the Cohen-Macaulay type of X_v at id.*

For $n \leq 4$, all X_w are Gorenstein, so these types are all equal to 1. For $n = 5$, the four non-Gorenstein Schubert varieties

$$X_{53241}, X_{35142}, X_{42513}, X_{52431}$$

all have type 2 at the identity. Using, `Schubsingular` we determined that X_{624351} is the unique Schubert variety having type 4 at the identity, while

$$X_{361542}, X_{426153}, X_{623541}, X_{625431}, X_{532614}, X_{632451}, X_{643251}$$

all have type 3 there. (We use the command `cansheafrank({2, 5, 0, 4, 3, 1}, {0, 1, 2, 3, 4, 5})` in `Schubsingular` to make the computation for X_{361542} and similar commands for the others.) The remaining Schubert varieties have type 1 or 2 at the identity. These can be distinguished by Theorem 6.6.

6.4. **Factoriality and the Class Group.** A variety is said to be **factorial** if the local ring at every point is a unique factorization domain. Since regular local rings are unique factorization domains, every smooth variety is factorial. Furthermore, all unique factorization domains are Gorenstein.

M. Bousquet-Mélou and S. Butler have characterized factorial Schubert varieties by the following theorem:

Theorem 6.13 ([8])**.** *The Schubert variety X_w is factorial if and only if w classically avoids 4231 and interval avoids $[3142, 3412]$.*

The considerations that led to Conjecture 6.7 also lead to the following conjecture.

Conjecture 6.14. *The order ideal $\mathcal{I}_{\text{not factorial}}$ is generated by the following families:*

(1) $\big[(a+1)a\cdots 1(a+b+2)\cdots(a+2), \quad (a+b+2)(a+1)a\cdots 2(a+b+1)\cdots(a+2)1\big]$ *for all integers $a, b > 0$.*
(2) $\big[(a+1)\cdots 1(a+3)(a+2)(a+b+4)\cdots(a+4), \quad (a+3)(a+1)\cdots 2(a+b+4)1(a+b+3)\cdots(a+4)(a+2)\big]$ *for all integers $a, b \geq 0$.*

As with the Gorenstein property, there is also an invariant which measures how far a local ring is from being a unique factorization domain, the (local Weil) class group. A local ring is a unique factorization domain if and only if the class group is trivial. We do not know of an algorithm to compute these class groups, or to otherwise check Conjecture 6.14.

6.5. **Multiplicity.** The **multiplicity** of a local ring $(R, \mathfrak{m}, \Bbbk)$ is the degree of the projective tangent cone $\mathrm{Proj}(\mathrm{gr}_\mathfrak{m} R)$ as a subvariety of the projective tangent space $\mathrm{Proj}(\mathrm{Sym}^* \mathfrak{m}/\mathfrak{m}^2)$. Equivalently, if the Hilbert-Samuel polynomial of R is $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then the multiplicity of R is $n! a_n$. Given a scheme X and a point p, the multiplicity of X at p, usually denoted $\mathrm{mult}_p(X)$, is the multiplicity of the local ring $(\mathcal{O}_{X_p}, \mathfrak{m}_p, \Bbbk)$.

It is an open problem to find a manifestly positive combinatorial rule for the multiplicity of X_v at the point e_u. Several such rules are known for the case where v is a Grassmannian permutation [21, 22, 24, 32]. The analogue of Corollary 6.3 for multiplicity is the following:

Corollary 6.15. *Suppose we have an interval pattern embedding of $[u, v]$ into $[x, w]$. Then* $\mathrm{mult}_{e_x}(X_w) = \mathrm{mult}_{e_u}(X_v)$.

Example 6.16. Ordinary pattern avoidance also fails for Corollary 6.15. With the same choice of u, v, w and Φ as in Example 6.4, we have $\mathrm{mult}_{e_{\Phi(u)}}(X_w) = 5$ while $\mathrm{mult}_{e_u}(X_v) = 2$.

We reformulate the problem of finding a rule for multiplicity in an incremental form as follows.

Problem 6.17. *Let $\mathcal{P}_k$ be the property "multiplicity of X_v is at least k". Find the generators for the ideal $\mathcal{I}_{\mathcal{P}_k}$ for all k.*

Now we look at how multiplicity can be computed. In our coordinates for $\mathcal{N}_{u,v}$, e_u is the point where $z_{ij} = 0$ for all i, j. In this case where the local ring is the localization of a ring S/J at the maximal ideal $\mathfrak{m}$ given by the vanishing of all the variables, the associated graded ring $\mathrm{gr}_\mathfrak{m} S/J$ is isomorphic to the localization of S/J' (at the maximal ideal given by the vanishing of all variables), where $J' = \langle f' \mid f \in J \rangle$, with f' defined to be the sum of all terms of minimal degree in f. To calculate the degree of J', we can then calculate a Gröbner basis of J'; this entire process of finding J' and finding a Gröbner basis for it can be accomplished in one step by finding the Gröbner basis (or an initial ideal) of J with respect to a term order that chooses a *lowest* degree term. Note that we are now using the grading where each variable has degree 1 rather than the grading discussed in section 4.

In `Macaulay 2`, we can simulate a term order choosing a lowest degree term by homogenizing the generators of $I_{u,v}$ using a new variable t (or by replacing the "1"s by "t"s in the matrix corresponding to $u\sigma u^{-1}(\Omega_u^\circ)$) and using a term order that refines the partial order by degree in t. We can then compute the initial ideal, send t to 1 and calculate the degree of the resulting (monomial) ideal to find multiplicity.

Example 6.18. Returning back to Example 3.4, let us calculate the multiplicity of X_{35142} at e_{13254}.

```
i2 : St  = QQ[t, z11, z12, z13,
             z21, z22, z23,
             z31,
             z41, MonomialOrder=>Eliminate 1]; n=5;
i4 : Rank = matrix({{1, 2, 3, 4 ,5},
                    {1, 2, 2, 3, 4},
                    {1, 2, 2, 3, 3},
                    {0, 1, 1, 2, 2},
```

```
                        {0, 1, 1, 1, 1}}); -- rank matrix of w
i5 : Gt   = matrix({{  t,    0,   0,   0, 0},
                     {z41,    0,   t,   0, 0},
                     {z31,    t,   0,   0, 0},
                     {z21, z22, z23,   0, t},
                     {z11, z12, z13,   t, 0}});
i6 : Jlist = trim(sum(flatten(for i from 0 to n-1 list
                              for j from 0 to n-1 list
                                minors(Rank_(n-i-1,j)+1,
                                  submatrix(Gt, {(n-i-1)..n-1}, {0..j}))))))
i7 : GBlist = gb(Jlist);
i8 : LTlist = leadTerm(gens(GBlist)); -- gives in(J_{v,w})
i9 : S    = QQ[z11, z12, z13,
               z21, z22, z23,
               z31,
               z41];
i10 : f = map(S, St, {1, z11, z12, z13, z21, z22, z23, z31, z41});
i11 : ELTlist = f(LTlist); -- gives in(J_{v,w}')
i12 : Dlist = degree(ideal(ELTlist))
o12 = 2
```

Hence $\mathrm{mult}_{e_{13254}}(X_{35142}) = 2$. (In Schubsingular, this calculation is automated by the command $\mathtt{mult(\{2,4,0,3,1\},\{0,2,1,4,3\})}$.) $\qquad\square$

One possible method for solving the problem of finding multiplicities would be to find a combinatorial description for the initial ideals resulting from the above algorithm, under a particularly good choice of term order. Under particularly good conditions, the set of pipe dreams for the matrix Schubert variety $\pi^{-1}(X_v)$ counts the multiplicity; it is shown by this method in [34] that the multiplicity of $X_{n23\cdots(n-1)1}$ at e_{id} is the Catalan number C_{n-2}, and conjectured there that this is the highest multiplicity (at any point) on any Schubert variety in $\mathrm{Flags}(\mathbb{C}^n)$.

We also have the analogues of Problems 6.11 and 6.12 for multiplicity.

Problem 6.19. *Characterize pairs (u, v) such that the multiplicity of X_v at e_u is larger than the multiplicity X_v at $e_{u'}$ for all u' with $u < u' \leq v$.*

Problem 6.20. *Find the multiplicity of X_v at id.*

Based on our calculations so far, the projection of the ideal for the property "multiplicity of X_v at e_u is at least 3" onto the second factor is an order ideal in the partial order given by ordinary pattern avoidance. Both geometric and combinatorial explanations of this phenomenon, if it indeed holds in general, would be interesting.

6.6. **Final remarks and summary for $n = 5$.** In this report we have discussed several semicontinuously stable invariants of Schubert varieties. We present a compact summary for $n = 5$, which can also be verified using Schubsingular.

Proposition 6.21.
- X_{52341} *is Gorenstein, has multiplicity 5 below 21354, multiplicity 1 where it is nonsingular, and multiplicity 2 everywhere else.*
- *The 4 non-Gorenstein X_w have multiplicity 3 and canonical sheaf rank 2 where singular.*

- *All other singular X_w have multiplicity 2 where singular.*

`Schubsingular` provides algorithms to compute such facts for larger n. However, already for $n = 6$ the situation is complex enough that we refer the reader to the software.

There are other interesting cases of invariants not considered here. As one example, a local ring is a **complete intersection** if it is the quotient of a regular local ring by a regular sequence. Remarkably, this is actually a homological property which is independent of the ambient ring. In our case, X_v is locally a complete intersection at e_u if and only if the first Betti number for $\mathcal{N}_{u,v}$ is equal to $\binom{n}{2} - \ell(v)$.

Not being locally a complete intersection is a semicontinuously stable property; indeed, the difference between the first Betti number of $\mathcal{N}_{u,v}$ and $\binom{n}{2} - \ell(v)$ is upper semicontinous. We conclude with the following question, which was raised independently by B. Hassett, R. Joshua and B. Sturmfels:

Problem 6.22. *Which Schubert varieties X_w are local complete intersections?*

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DEPARTMENT OF MATHEMATICS, MATHEMATICAL SCIENCES BUILDING, ONE SHIELDS AVE., UNIVERSITY OF CALIFORNIA, DAVIS, CA, 95616, USA

E-mail address: awoo@math.ucdavis.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455, USA; DEPARTMENT OF STATISTICS *and* THE FIELDS INSTITUTE, UNIVERSITY OF TORONTO, TORONTO, ONTARIO, M5T 3J1, CANADA

E-mail address: ayong@math.umn.edu, ayong@fields.utoronto.ca